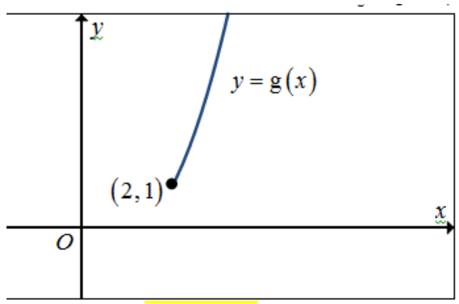
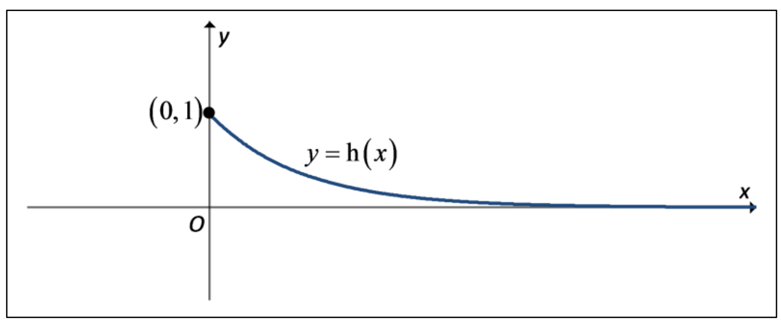
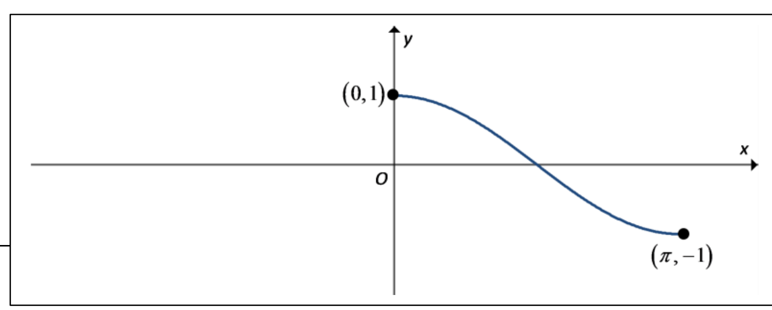
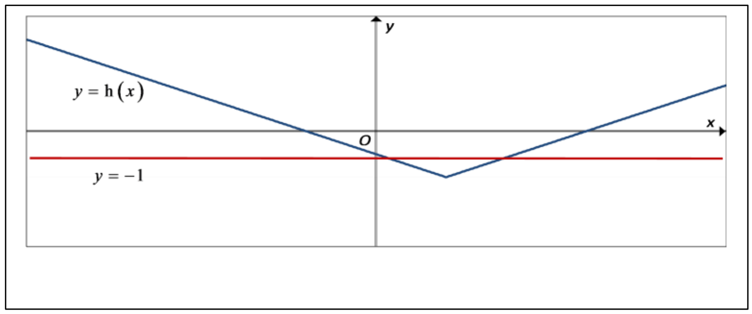
Skill Sets

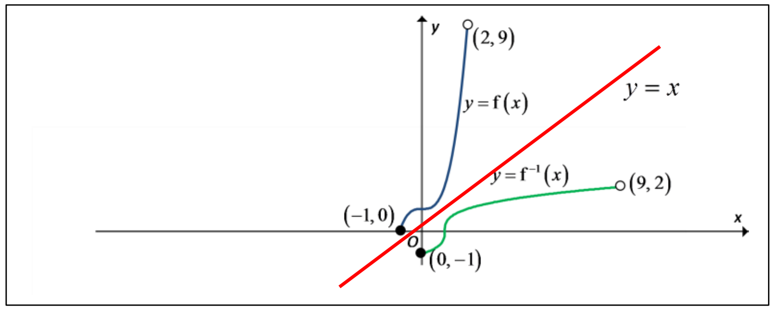
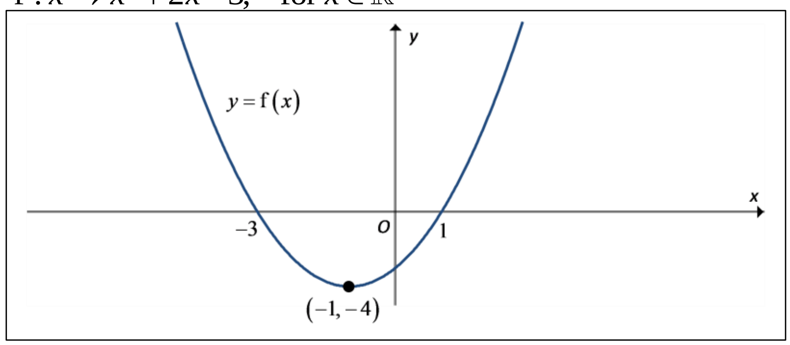
Chapter 5 Functions

No.	Skills	Examples of questions involving the skills
1.	Sketch the graph of the function according to the domain. - Students tend to sketch the graph without referring to the domain given by the function.	**(Lecture Notes Example 2 (b))** $\mathrm{g}: x \to x^2 - 2x + 1, \quad x \in \mathbb{R}, x \geq 2$ 
2.	Find the range of the function by drawing good graphs. - Students must realize that the accuracy of the range found depends on accurate graphs. Do pay attention to any asymptotes, turning points and other features of the graph.	**(Lecture Notes Example 2 (c))** $\mathrm{h}: x \to \mathrm{e}^{-2x}, \quad x \in \mathbb{R}, x \geq 0$  $R_{\mathrm{h}} = (0,1]$ Here the horizontal asymptote is an important feature for figuring out the range.
3.	Determine if the function is one to one. Method 1: By applying the 'Horizontal Line Test' and stating the correct reason to justify whether the function is one to one.	<u>When the function is one to one:</u> **(Lecture Notes Example 3(b))** $\mathrm{g}: x \to \cos x, \quad x \in \mathbb{R}, 0 \leq x \leq \pi$, 

		Every horizontal line intersects the graph of $y = \mathrm{g}(x)$ at no more than one point, thus the function g is a one to one function. Therefore its inverse exists. <u>When the function is not one to one:</u> **(Lecture Notes Example 3(c))** $\mathrm{h}: x \mapsto \lvert x-1 \rvert - 2, \quad x \in \mathbb{R}$.  Draw a horizontal line on your graph to show that it intersects the curve at two points. Thus h is not a one to one function. **OR** Give a counterexample like $x = -1, x = 3$ gives the same *y* value = 0.
4.	Find the rule of the inverse function by letting $y = \mathrm{f}(x)$ and making x as the subject eventually. (a) When $\mathrm{f}(x)$ is a quadratic function. <u>Method 1:</u> Complete square <u>Method 2:</u> Express the quadratic expression in the form of	**(Lecture Notes Example 5)** $\mathrm{f}: x \to x^2 + 2x - 3, \quad \text{for } x \in \mathbb{R}$. From $y = \mathrm{f}(x)$, we have $y = x^2 + 2x - 3$ $\Rightarrow y = (x+1)^2 - 4$ $\Rightarrow x = -1 \pm \sqrt{y+4}$. Since $x \geq -1$, therefore we take $x = -1 + \sqrt{y+4}$. Thus $\mathrm{f}^{-1}(y) = -1 + \sqrt{y+4} \Rightarrow \mathrm{f}^{-1}(x) = -1 + \sqrt{x+4}$. From $y = \mathrm{f}(x)$, we have $y = x^2 + 2x - 3$

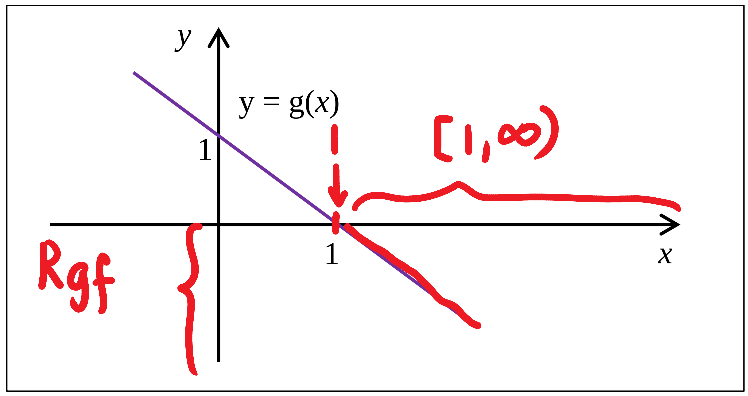
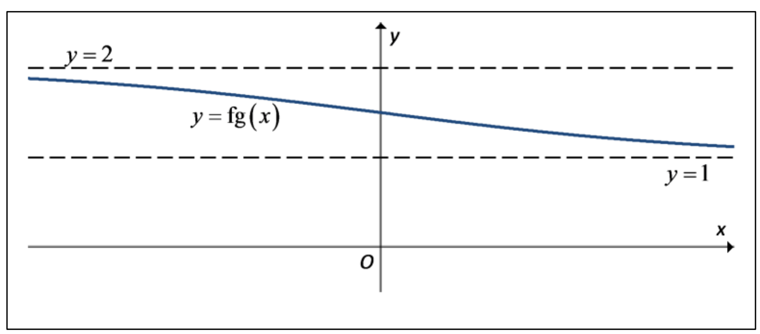
	$ax^2+bx+c=0$ and find x in terms of y using $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.	$\Rightarrow x^2+2x-3-y=0$ $\Rightarrow x=\dfrac{-2\pm\sqrt{4-4(1)(-3-y)}}{2}$ $\Rightarrow x=\dfrac{-2\pm\sqrt{4(4+y)}}{2}$ $\Rightarrow x=-1\pm\sqrt{4+y}$ Since $x\ge -1$, therefore we take $x=-1+\sqrt{y+4}$. Thus $\mathrm{f}^{-1}(y)=-1+\sqrt{y+4}\Rightarrow \mathrm{f}^{-1}(x)=-1+\sqrt{x+4}$.
	(b) When f(x) is a logarithmic function - Realize that inverse of a ln function is the exponential function.	**(Tutorial Q1(d))** $\mathrm{f}: x\mapsto -\ln 2x,\quad x\in\mathbb{R},\ x>0$ Let $y=-\ln 2x$ $-y=\ln 2x\Rightarrow \mathrm{e}^{-y}=2x$ [Note: $\mathrm{e}^{\ln 2x}=2x$] $\Rightarrow x=\dfrac{1}{2}\mathrm{e}^{-y}$ $\therefore \mathrm{f}^{-1}(x)=\dfrac{1}{2}\mathrm{e}^{-x}$.
	(c) When f(x) is a modulus function - Resolve modulus function into 2 expressions. i.e. $\lvert x-1\rvert=(x-1)$ or $-(x-1)$	**(Tutorial Q1(e))** $\mathrm{f}: x\mapsto \left\lvert\dfrac{2}{x-1}\right\rvert,\quad x\in\mathbb{R},\ x<1$ Let $y=\left\lvert\dfrac{2}{x-1}\right\rvert$ Then, $y=-\dfrac{2}{x-1}$ since $x<1$. $-y=\dfrac{2}{x-1}\Rightarrow -\dfrac{1}{y}=\dfrac{x-1}{2}\Rightarrow x=1-\dfrac{2}{y}$ $\therefore \mathrm{f}^{-1}(x)=1-\dfrac{2}{x}$.
5.	Find the domain and range of the inverse function using the relation:	Domain of f^{-1} = Range of f Range of f^{-1} = Domain of f

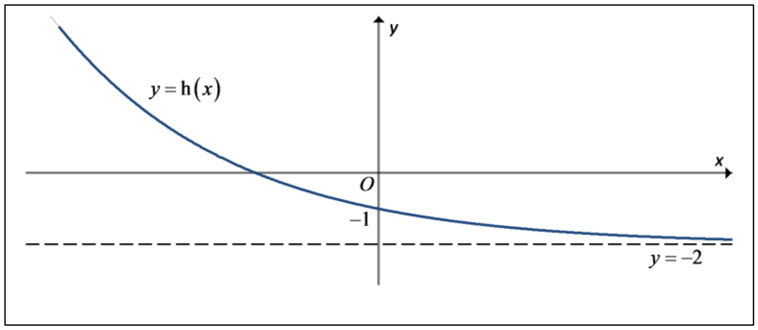
<table>
<tr><td>6.</td><td>Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram
Important to note:
(a) change of coordinates from (x, y) to (y, x) after reflecting the graph of $y = \mathrm{f}(x)$ in the line $y = x$,

(b) same scale on both axes.</td><td>(Lecture Notes Example 4)

</td></tr>
<tr><td>7.</td><td>Restrict the domain of f so that f^{-1} exists.</td><td>(Lecture Notes Example 5(i))
$\mathrm{f} : x \to x^2 + 2x - 3, \quad \text{for } x \in \mathbb{R}$


From the graph, we observe that for the inverse to exist, i.e. for f is one to one, the largest domain we can go for is $x \geq -1$.
Thus the smallest value of p is -1.</td></tr>
<tr><td>8.</td><td>State the range of f and domain of g to check the existence of composite function gf.</td><td>Students need to check $R_{\mathrm{f}} \subseteq \mathrm{D}_{\mathrm{g}}$ for gf exists.
Students must state the range of f and domain of g explicitly before concluding if the former is a subset of the latter.</td></tr>
<tr><td>9.</td><td>Find the rule of composite function gf and its domain</td><td>(Extension of lecture notes example 6)
$\mathrm{f} : x \to 1 + \sqrt{x}, \ x \in \mathbb{R}, x \geq 0$ and $\mathrm{g} : x \to 1 - x, \ x \in \mathbb{R}$
$\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x)) = \mathrm{g}(1 + \sqrt{x}) = 1 - (1 + \sqrt{x}) = -\sqrt{x}$

$D_{\mathrm{gf}} = D_{\mathrm{f}} = [0, \infty)$</td></tr>
</table>

10.	Find the range of gf using Method 1: Do it stages. $D_{gf} = D_f \xrightarrow{f} R_f \xrightarrow{g} R_{gf}$	**(Lecture Notes Example 6)** $f : x \to 1+\sqrt{x},\ x \in \mathbb{R}, x \geq 0$ and $g : x \to 1-x,\ x \in \mathbb{R}$ $D_{gf} = D_f = [0,\infty) \xrightarrow{f} R_f = [1,\infty) \xrightarrow{g} ?$ Use the graph of y = g(x), substitute $[1,\infty)$ into the function g.  $R_{gf} = (-\infty, 0]$
	Method 2: Graphical method - The graph of fg may not be easy to draw accurately. - This method is recommended if fg is a simple function.	**(Lecture Notes Example 7(iii))** Sketch the graph of y = fg(x) according to the domain of g. Find the range of fg. $y = \text{fg}(x) = 1 + \frac{1}{e^x + 1}, \quad x \in \mathbb{R}$. Sketch the graph of the composite function fg subjected to its domain.  From the graph, we obtain the range of $\text{fg} = (1, 2)$.

11.	Restrict the domain of g so that $R_g \subseteq D_f$ for composite function fg to exist. - Basically we crop the function of h so that the new $R_h \subseteq D_g$.	**(Lecture Notes Example 8)** $g: x \to \ln(x+1), \quad x \in \mathbb{R}, x > -1,$ $h: x \to e^{-x} - 2, \qquad x \in \mathbb{R}.$  $R_h = (-2, \infty)$, $D_g = (-1, \infty)$ Since $R_h \not\subset D_g$, gh does not exist. For gh to exist, $R_h \subseteq (-1, \infty)$. The greatest possible range of $h = (-1, \infty)$. From the graph, the greatest possible domain of h for gh to exist is $(-\infty, 0)$.
12.	Solving the equation f(x) = f^{-1} (x) is the same as solving f(x) = x.	**(Tutorial Q2(iii))** Since the graphs of y = f(x), $y = f^{-1}(x)$ and $y = x$ will meet at the same point, to find the exact solution of the equation f(x) = $f^{-1}(x)$, we could solve f(x) = x. i.e., $3x^2 - 1 = x$.
13.	Know the difference between the graph of $f^{-1}f$ and $f f^{-1}$	**(Tutorial Q6(iii))** Given $f: x \mapsto (x-1)^2 + 2, \ x \in \mathbb{R}, x < 1$ What is the difference between the graph of $f^{-1}f$ and $f f^{-1}$? $f^{-1}f(x) = f f^{-1}(x) = x$ [This result is true all the time, regardless of f]. Therefore $f^{-1}f$ and $f f^{-1}$ share the same rule. However, the domain of $f^{-1}f$ = domain of f = $(-\infty, 1)$ while domain of ff^{-1} = domain of $f^{-1} = (2, \infty)$. You should also realize that are composite functions $f^{-1}f$ and $f f^{-1}$ too and they always exist. Check the condition for

		the existence of composite function yourself.
14.	You should also use the summary on Page 20 to help you revise through the topic.	
15.	The learning objectives at the first page of the lecture notes is also good for knowing the important concepts that are taught in this chapter.	